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LETTER TO THE EDITOR

Scaling of distribution eigenvectors in a 1D Anderson model

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Abstract. It is shown numerically that the distribution of squared components of eigenvectors of the Anderson 1D tight binding equation on lattices of finite lengths, is parametrized by the single scaling parameter $x = \xi_\infty/N$, where ξ_∞ is the localization length for the infinite lattice and N is the number of sites of the finite lattice.

The 1D Anderson model describes the conductance properties of wires with impurities at very low temperatures. It is defined by a tight binding equation with random potential

$$\psi_{n+1} + \psi_{n-1} + V_n\psi_n = E\psi_n \tag{1}$$

where the strenghts V_n of the potential at different sites, are independent random variables with the same uniform distribution in $[-W/2, W/2]$, and E is the energy. The main property of this model is the exponential localization of eigenfunctions, which occurs at any non-zero disorder W and is usually measured by a length ξ_∞, defined as the inverse of the Lyapounov exponent

$$\gamma_\infty(E, W) = -\lim_{N\to\infty}(1/N)\langle\ln|\psi_N|\rangle \tag{2}$$

averaged over different realizations of disorder. The Lyapounov exponent is easily computed by the technique of transfer matrices, which is efficient for not too small disorder, a case which is more conveniently treated with asymptotic expansions [1].

When samples of finite length $1 \leqslant n \leqslant N$ are considered, it is found that the ratio ξ_N/N of the localization length to the sample length is a function of the single scaling parameter $x = \xi_\infty/N$. This property corresponds to the 1D case of the general scaling theory of conductance for disordered systems [2]. There are several ways to characterize a localization length for a finite lattice [3, 4]. The following ones are based on generalized entropies [5]:

$$\xi^{(1)} = \exp\left[-\left\langle\sum_{n=1}^{N}|\psi_n|^2\log|\psi_n|^2\right\rangle\right] \qquad \xi^{(q)} = \left\langle\sum_{n=1}^{N}|\psi_n|^{2q}\right\rangle^{1/1-q} \qquad (q \geqslant 2) \tag{3}$$

which are related to the moments of the distribution of squared components

$$p(y) = \left\langle\frac{1}{N}\sum_{n=1}^{N}\delta(y - N|\psi_n|^2)\right\rangle. \tag{4}$$

The lengths for $q = 1, 2$ are respectively the information length and the participation ratio, both commonly used. For the model (1) without disorder ($W = 0$) the lengths are easily evaluated

$$\xi_0^{(1)} = 2\mathrm{e}^{-1}(N+1) \qquad \xi_0^{(q)} = (N+1)\left[\frac{1}{2^q}\binom{2q}{q}\right]^{1/1-q}. \tag{5}$$

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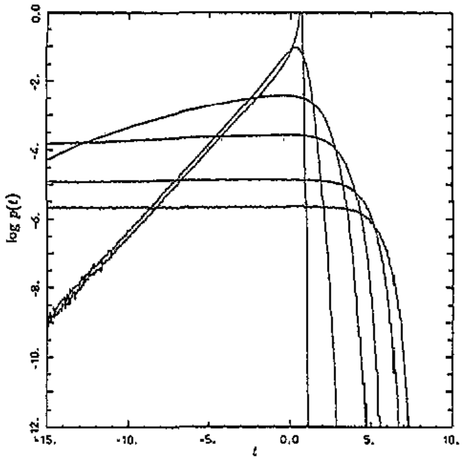

Figure 1. The log of the probability density of $t = \log y$ for eigenvectors in the energy window $0.05 < E < 0.15$, lattice size $N = 3200$ and six values of disorder. From rigth to left on horizontal axis: W=3.00, 2.00, 1.00, 0.50, 0.13, 0.016. The quantity y is defined in (7).

Since the free model corresponds to the most extended states, it is convenient to introduce the ratios $\beta_q = \xi^{(q)}/\xi_0^{(q)}$, with values from 1 to 0 as disorder increases. The case of $q = 1$ has been investigated in [6], and a very simple scaling relation was obtained:

$$\beta_1 = \frac{cx}{1+cx} \qquad (c \approx 2.8). \tag{6}$$

The same form of scaling relation, here valid for tridiagonal matrices with diagonal disorder, was previously found in the theory of band random matrices (BRM) [7] and then derived analytically for the case $q = 2$ [8]. BRM were first introduced by Wigner, and have been recently investigated in detail as models for chaotic hamiltonians or intermediate level statistics. A BRM ensemble is defined as the set of $N \times N$ symmetric matrices with non-zero matrix elements given by independent and identically distributed gaussian random variables, restricted in a band of width $2b - 1$. For $N \to \infty$ the eigenstates are localized with length ξ_∞ proportional to b^2. [9]. For finite matrices the maximally extended states correspond to $b = N$, the case of Gaussian orthogonal ensemble. It was numerically found that localization ratios β_q, and separation of neighbouring eigenvalues depend on the single parameter b^2/N [10, 11]. Finally, Zyczkowski *et al* [12] have shown numerically that the whole distribution of squared components of eigenvectors are parametrized by the same single parameter. This result has been recently analytically derived [13].

In analogy with the above result, in this letter it is shown that the distribution of squared components of eigenvectors fulfils scaling also in the Anderson model, with the scaling parameter given by ξ_∞/N.

Since localization depends on energy, we restrict to a definite energy window $E_0 - \Delta < E < E_0 + \Delta$. The eigenvectors corresponding to eigenvalues in the window are then computed for several matrices with given size N and disorder parameter W. A huge collection of values ψ_n^2 is easily assembled, and since the normalization of eigenvectors forces components to scale as $N^{-1/2}$, we introduce the variable

$$y = N\psi_n^2. \tag{7}$$

It will be convenient, however, to investigate the distribution in the variable $t = \log y$.

Before examining scaling, let us see qualitatively how the distribution $p(y)$ depends on disorder, for a given value N. In figure 1 we plot $\log p(t)$ for $N = 3200$, $E = 0.1 \pm 0.05$ and several values of W. For low and high disorder, the small t behaviour of $\log p(t)$ appears to be linear, with slopes $1/2$ and zero; this is easily explained as follows. For zero disorder

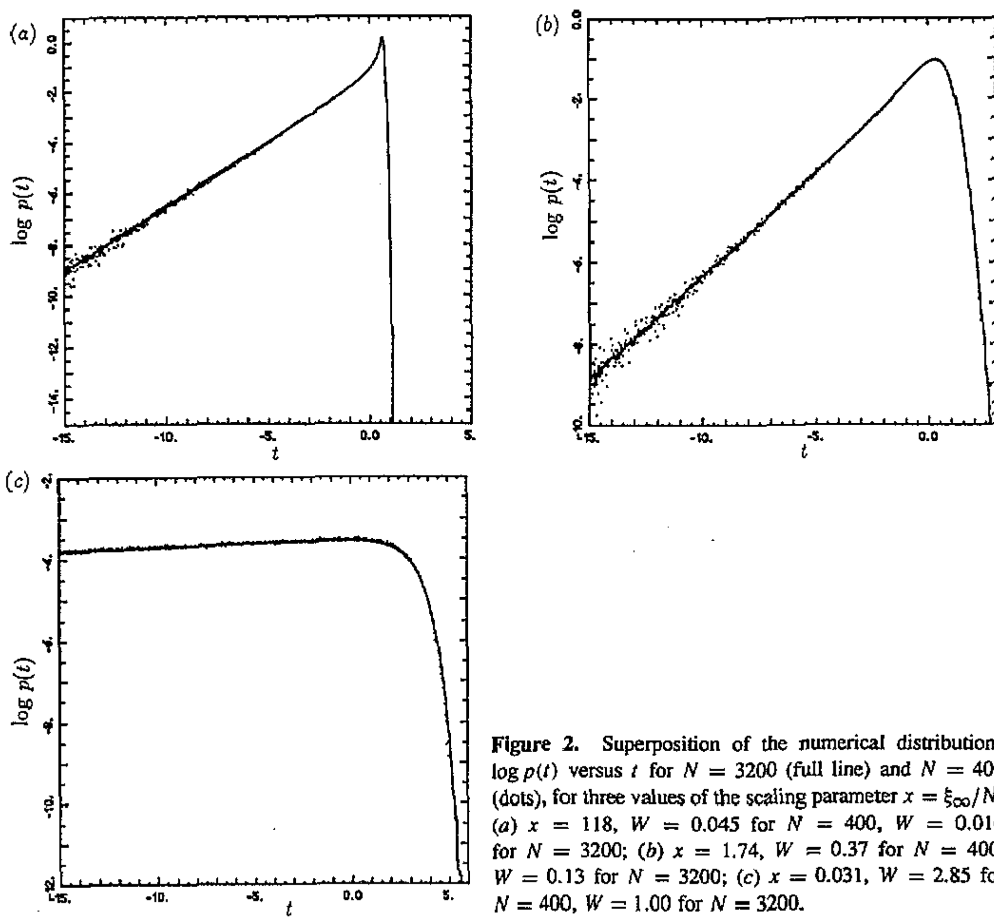

Figure 2. Superposition of the numerical distributions $\log p(t)$ versus t for $N = 3200$ (full line) and $N = 400$ (dots), for three values of the scaling parameter $x = \xi_\infty/N$. (*a*) $x = 118$, $W = 0.045$ for $N = 400$, $W = 0.016$ for $N = 3200$; (*b*) $x = 1.74$, $W = 0.37$ for $N = 400$, $W = 0.13$ for $N = 3200$; (*c*) $x = 0.031$, $W = 2.85$ for $N = 400$, $W = 1.00$ for $N = 3200$.

the equation (1) is solved by plane waves. The boundary conditions $\psi_0 = \psi_{N+1} = 0$ select the eigenvalues $E_k = 2\cos[2k\pi/(N+1)]$, $k = 1 \ldots N$, with corresponding eigenvector

$$\psi_n^{(k)} = \sqrt{2/(N+1)}\sin(nk\pi/(N+1)). \tag{8}$$

The distribution of the variable y, with support $0 < y < 2$ is easily calculated:

$$p(y) = (1/\pi)1/\sqrt{y(2-y)}. \tag{9}$$

In the variable t this would give the slope $1/2$, which persists for small disorder since, for finite N, perturbation theory is meaningful. In the regime of strong disorder, the localization length $\xi_\infty(W)$ becomes much smaller than the size N of the lattice and the tails of the eigenfunctions are well described by $\psi_n = A\exp(-|n-n_0|/\xi_\infty)$. This implies that for small y the density is

$$p(y) = \left(\frac{\xi_\infty}{N}\right)\frac{1}{y} \tag{10}$$

which corresponds in figure 1 to horizontal lines with height $h = \log(\xi_\infty/N)$. For the cases represented in figure 1 one actually computes by means of transfer matrices: for $W = 3$ $h = -5.65$, for $W = 2$ $h = -4.83$, for $W = 1$ $h = -3.46$. These values agree quite well with figure 1.

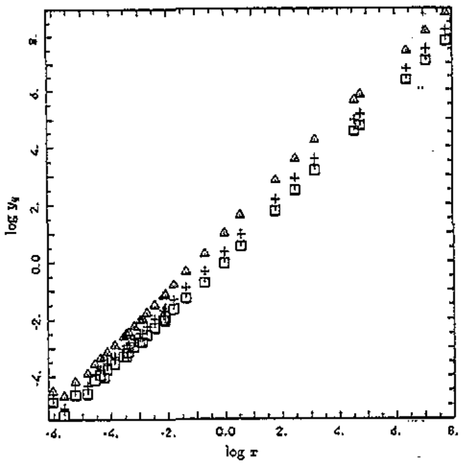

Figure 3. Scaling of normalized lengths $\beta^{(q)} = \xi^{(q)}/\xi_0^{(q)}$. The quantities $\log(y_q) = \log(\beta^{(q)}/1 - \beta^{(q)})$ are plotted against $\log x$ for $q = 1$ (triangle), $q = 2$ (cross) and $q = 3$ (squares), superposing data for $N = 400$ and $N = 3200$.

If these computations are done for different sizes and disorder parameters, such that the ratio ξ_∞/N is the same, the probability densities $p(t)$ are found to overlap. We considered the two sizes $N = 400$ and $N = 3200$, collecting eigenvectors in the same energy window as above, for a statistical sample of 800 and 100 matrices respectively. In figure 2 we give examples of overlapping distributions, with the scaling parameter $x = \xi_\infty/N$ taking the values (*a*) 118 (very delocalized regime), (*b*) 1.74 (intermediate) and (*c*) 0.031 (localized regime). The scaling has been checked up to $x = 0.002$. The plotted curve is that of $\log p(t)$, versus t, $t = \log y$. In all cases the curves overlap very well, and would even better for a more accurate choice of the values of the disorder parameter W, yielding closer values of the scaling parameter x. The scaling in x of the distributions of squared components obviously implies that of the localization lengths (3). In figure 3 the ratios $\beta^{(q)}$ are plotted for $q = 1, 2, 3$. The plot indicates an approximate linear behaviour which implies a relation of type (6): $\beta^{(q)} = c_q x^{\alpha_q}/(1 + c_q x^{\alpha_q})$, with $c_1 = 2.85$ and $\alpha_1 = 1.00$, $c_2 = 1.62$ and $\alpha_2 = 0.97$, $c_3 = 1.16$ and $\alpha_3 = 0.96$

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